A Filtering Formula for the Conditional Intensity of the Renewal Hawkes Process

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A renewal Hawkes process consists of two subprocess: the background subprocess is a renewal process, and the clustering subprocess contains the offspring produced by all previous events (background or clustering). This process is the same as the Hawkes self-exciting process except that the background process is changed from a Poisson process to a renewal process.

Suppose the observation of the point process is $\{(t_i, m_i) : i = 1, 2, \cdots, n\}$ in a given time interval $[0, T]$, with m_i taking values 1 (background event) or 0 (excited event). Denote by $\mathcal{C}_t$ the Complete history of the renewal Hawkes process N up to time t but not including t, in which whether an event is a background event or not is always known. The conditional intensity of the complete Hawkes process is

$$\lambda_{N_C}(t, m \mid \mathcal{C}_t) = \mathbf{1}(m = 1)\, \varphi(t \mid \mathcal{C}_t) + \mathbf{1}(m = 0)\, \xi(t \mid \mathcal{C}_t) \tag{1}$$

with $\varphi(t \mid \mathcal{C}_t) = \mu(t - t_{\Phi(t_-)})$ and $\xi(t \mid \mathcal{C}_t) = \sum_{i=1}^{N(t_-)} g(t - t_i) = \xi(t \mid \mathcal{H}_t)$. The stochastic intensity of the ground renewal Hawkes process is

$$\lambda_N(t \mid \mathcal{C}_t) = \varphi(t \mid \mathcal{C}_t) + \xi(t \mid \mathcal{C}_t) = \mu(t - t_{\Phi(t_-)}) + \sum_{i=1}^{N(t_-)} g(t - t_i). \tag{2}$$

In most cases, the marks are unknown (missing from observation), we need to known the stochastic condition intensity, $\lambda_N(t \mid \mathcal{H}_t)$ or briefed as $\lambda(t \mid \mathcal{H}_t)$, of the ground RH process conditional on the observation history $\mathcal{H}_t$. Since $\mathcal{H}_t \subseteq \mathcal{C}_t$,

$$\lambda(t \mid \mathcal{H}_t) = \mathbf{E}\left[\lambda(t \mid \mathcal{C}_t) \mid \mathcal{H}_t\right] = \mathbf{E}\left[\, \mu(t - t_{\Phi(t_-)}) \,\middle|\, \mathcal{H}_t\right] + \sum_{i=1}^{N(t_-)} g(t - t_i)$$

By definition, $\varphi(t \mid \mathcal{C}_t) = \varphi(t \mid \Phi(t_-)$ where $\Phi(u)$ is the identification of the last background event not later than u. Thus,

$$\lambda(t \mid \mathcal{H}_t) \quad = \quad \sum_{i=1}^{N(t_-)} \mu(t - t_i)\ \Pr\{\Phi(t_-) = i \mid \mathcal{H}_t\} + \sum_{i=1}^{N(t_-)} g(t - t_i) \tag{3}$$

gives the stochastic intensity of the process conditioning on $\mathcal{H}_t$.

By using the Bayesian formula, we can obtain that $\varphi(t \mid \mathcal{H}_t)$ satisfies the recursive equation

$$\varphi(t \mid \mathcal{H}_t) = \sum_{i=1}^{N(t_-)} \left(\mu(t - t_i)\, \mathrm{e}^{-\int_{t_i}^{t} [\mu(u - t_i) - \varphi(u \mid \mathcal{H}_u)]\, \mathrm{d}u} \, (1 - \rho_i) \prod_{k=i+1}^{N(t_-)} \rho_k \right). \tag{4}$$

where

$$\rho_k \equiv \frac{\xi(t_k \mid \mathcal{H}_{t_k})}{\lambda(t_k \mid \mathcal{H}_{t_k})} = \frac{\lambda(t_k \mid \mathcal{H}_{t_k}, \Phi(t) = i)}{\lambda(t_k \mid \mathcal{H}_{t_k})}, \qquad \text{for } t_i < t_k < t \tag{5}$$

is the probability that event k is not from background, knowing the history of the ground RH process up to time t_k.

The above results are compared to the filtering formulae obtained by Chen and Stindl (2018) and applied to a simulated dataset and a dataset selected from the JMA earthquake catalog.

References:

Chen, F. & Stindl, T. (2018) *Direct Likelihood Evaluation for the Renewal Hawkes Process*. **Journal of Computational and Graphical Statistics**, 27:1, 119-131. DOI: 10.1080/10618600.2017.1341324

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