The relationship between Dickey-Fuller test and Sequential unit root test for first-order autoregressive model

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Introduction

We consider unit root tests against local alternative when a first-order autoregressive (AR(1)) process is observed sequentially. We derive the relationship between the sequential unit root test (SURT) and the Dickey-Fuller (DF) test by simulation and theoretical study, when both tests have the same power under the presumed alternative hypothesis. The joint asymptotic density of the sequential coefficient statistics and the stopping times are derived by using theory of Bessel bridges in Pitman and Yor(1982).

Model and result

We consider a scale AR(1) process: $x_n = \beta x_{n-1} + \epsilon_n, n = 1, 2, \cdots$, with $\epsilon_n \sim i.i.d. \left(0, \sigma^2\right)$ and initial value $x_0 = 0$. In sequential unit root test, suppose we observe $x_0, x_1, x_2, \ldots$ sequentially and propose to stop sampling at time $\tau_c = \inf \left\{N : \sum_{n=1}^{N} x_{n-1}^2 \geq \sigma^2 c\right\}$, where c is the predetermined. The alternative hypotheses of DF and SURT are $H_1 : \beta = 1 - \theta/N$ and $H_1 : \beta = 1 - \delta/\sqrt{c}$ respectively. As $c \to \infty$, the limit of test statistics of SURT is

$$\sqrt{c}\left(\hat{\beta}_{\tau_c} - 1\right) \to \begin{cases} N(0,1) & under \quad H_0 \\ N(-\delta, 1) & under \quad H_1 \end{cases}$$

while the limit of DF's coefficient test statistics is

$$N\left(\hat{\beta}_N - 1\right) \to \begin{cases} \int_0^1 W_s dW_s / \int_0^1 W_s^2 ds & under \quad H_0 \\ \int_0^1 X_s dX_s / \int_0^1 X_s^2 ds & under \quad H_1 \end{cases}$$

where W is a Brownian motion and X is an Ornstein=Uhlenbeck process; $dX_t = -\theta X_t dt + dW_t$.

Let Ψ_θ and Φ be the distribution function of DF test statistics and standard normal distribution respectively. The powers of DF and SURT are $\Psi_\theta\left(\Psi_0^{-1}(\alpha)\right)$ and $\Phi\left(\Phi^{-1}(\alpha) + \delta\right)$ for significant level α. By simulation and theoretical study, we found that SURT has the same power with DF test when $\delta \approx K(\alpha)\theta$. Letting $\phi(z) = \Phi'(z)$,

$$K(\alpha) = E[1\{\int_0^1 WdW/\int_0^1 W^2 ds \leq \Psi_0^{-1}(\alpha)\}(-\int_0^1 WdW)]/\phi(\Phi^{-1}(\alpha))$$

is a number dependent on α, for example $K(0.05) = 0.180874$. Hence,we conclude that

$$\Psi_\theta\left(\Psi_0^{-1}(\alpha)\right) \approx \Phi\left(\Phi^{-1}(\alpha) + K(\alpha)\theta\right)$$

which gives a simple computation method of the local powers of DF test.

By simulation and theoretical study, we also found that under the null hypothesis, both SURT and DF obtain the true size asymptotically, and SURT has a smaller expected sample size than DF.

Reference

1. Nabeya, S. and Tanaka. K. (1990). Limiting power of unit root tests in time series regression, *J. Econometrics* **46** 247-271.
2. Nagai, K., Nishiyama, Y. and Hitomi, K. (2018). Sequential test for unit root in AR(1) model, *KIER Discussion Paper*, No.1003, Kyoto University
3. Pitman and Yor(1982). A Decomposition of Bessel Bridges. Z.W 59, 425-457