Information Geometry of Complex Autoregressive Models and its Positive Superharmonic Priors

Hidemasa Oda and Fumiyasu Komaki

Graduate School of Information Science and Technology, the University of Tokyo

It is difficult to decide what are the good features to be equipped on statistical manifolds. The Kähler structure on a statistical manifold is considered to be one of the good properties that the statistical model should have. Complex-valued stochastic processes are useful models for parametrizing complex or bivariate signals. Choi and Mullhaupt showed that the information geometry of complex signals is Kähler [2]. By making use of this structure, they showed that, for the complex autoregressive models of order 2, there is a Bayesian predictive distribution which asymptotically dominates the Bayesian predictive distribution based on the Jeffreys prior. We generalized this theorem to the complex autoregressive models of any order p.

Let us consider a stable discrete-time linear system with a complex scalar input and a complex scalar output. Let $h = \{h_t\}$ be a complex linear operator that maps an input $x = \{x_t\}$ to an output $y = \{y_t\}$ defined by $y_t = \sum_i h_i x_{t-i}$ where all x_t, y_t and h_t are complex numbers. The output y is the convolution $y = h * x$ of the input x and the operator h. We assume throughout this presentation that the linear system is invertible and that both the original linear system and the inversed linear system are stable and causal. We call the system real if all h_t are real numbers. The function $H(z) := \sum_t h_t z^{-t}$ of a formal variable z is called the transfer function of the linear system, and the function $S(\omega) := \left|H(e^{i\omega})\right|^2$ is called the power spectral density of the linear system. If the linear system is real, then S is an even function on $[-\pi, \pi]$. The α-geometry of the statistical manifold of real autoregressive models was investigated in [1].

Let $\mathrm{AR}(\sigma^2; p; \mathbb{C}) := \left\{ S(\omega) = \sigma^2 \left| \frac{1}{\prod_{j=1}^{p}(1-\xi^j e^{-i\omega})} \right|^2 \,\middle|\, |\xi_j| < 1 \right\}$ be the set of all the complex autoregressive models of order p. Then, the statistical manifold $\mathrm{AR}^\circ(\sigma^2; p; \mathbb{C}) := \{S \in \mathrm{AR}(\sigma^2; p; \mathbb{C}) \mid |g| \neq 0\}$ is a Kähler manifold, where $g_{i\bar{j}} := \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(\partial_i \log S(\omega)\right)\left(\partial_{\bar{j}} \log S(\omega)\right) d\omega$ is the Fisher information matrix of the complex-valued stochastic process. We have investigated the Bayesian predictive distribution on the Kähler manifold $\mathrm{AR}^\circ(\sigma^2; p; \mathbb{C})$. We have proved that the function $\phi = \prod_{k,l}(1 - \xi^k \bar{\xi}^l)$ is positive and superharmonic $\Delta\phi = -2p(p+1)\phi$ on $\mathrm{AR}^\circ(\sigma^2; p; \mathbb{C})$. The $p = 2$ case has already been proved in [2]. For the real case $\mathrm{AR}^\circ(\sigma^2; p; \mathbb{R})$, the similar result has also been given for $p = 2$ in [3], and for general p in [4]. We have proved that the prior $\pi_H := \phi\, \pi_J$ asymptotically dominates the Bayesian predictive distribution based on the Jeffreys prior π_J,

$$E_S\left[D_{\mathrm{KL}}\left(S \,\middle\|\, \hat{S}_{\pi_J}\right) \,\middle|\, \xi\right] - E_S\left[D_{\mathrm{KL}}\left(S \,\middle\|\, \hat{S}_{\pi_H}\right) \,\middle|\, \xi\right] = \frac{1}{N^2}\left(2\|\xi\| + 2p(p+1) + o(1)\right) \quad ,$$

where $\|\xi\| := 2 g_{i\bar{j}} \xi^i \bar{\xi}^j = \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\xi^i \bar{\xi}^j}{1 - \xi^i \bar{\xi}^j}$.

References

[1] Shun-ichi Amari. Differential geometry of a parametric family of invertible linear systems—riemannian metric, dual affine connections, and divergence. *Mathematical systems theory*, 20(1):53–82, Dec 1987.

[2] Jaehyung Choi and Andrew P. Mullhaupt. Kählerian information geometry for signal processing. *Entropy*, 17(4):1581–1605, 2015.

[3] Fuyuhiko Tanaka and Fumiyasu Komaki. A superharmonic prior for the autoregressive process of the second-order. *Journal of Time Series Analysis*, 29(3):444–452.

[4] Fuyuhiko Tanaka and Fumiyasu Komaki. Asymptotic expansion of the risk difference of the bayesian spectral density in the autoregressive moving average model. *Sankhya A*, 73(1):162–184, Feb 2011.