Compatibility and Attainability of Matrices of Correlation-based Measures of Concordance

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Measures of concordance (MOC) have been widely used to summarize non-linear dependence among random variables. For more than two variables, matrices of pairwise MOCs are of interest. *Compatibility* concerns whether a given square matrix can be realized as a matrix of MOCs of some random vector, and *attainability* asks how to construct such a random vector. Results related to these concepts are present in the talk based on Hofert and Koike (2019).

For a copula $(U_1, U_2) \sim C$ and Pearson's correlation coefficient ρ, we consider a bivariate measure of the form

$$\kappa_{g_1,g_2}(U_1, U_2) = \rho\big(g_1(U_1), g_2(U_2)\big), \quad g_1, g_2 : [0,1] \to \mathbb{R} : \text{ measurable.} \tag{1}$$

For what g_1, g_2 does κ_{g_1,g_2} become a MOC (in the sense of Scarsini, 1984)? We will show that if $g_1 : [0,1] \to \mathbb{R}$ and $g_2 : [0,1] \to \mathbb{R}$ are continuous functions, then g_1 and g_2 must be both increasing or both decreasing, that is,

$$(g_1(u') - g_1(u))(g_2(v') - g_2(v)) \geq 0, \quad 0 \leq u < u' \leq 1, \quad 0 \leq v < v' \leq 1.$$

for κ_{g_1,g_2} defined in (1) to be a MOC. Next, if (1) is of the form $\kappa_{G_1^{-1},G_2^{-1}}$ for two distributions G_1, G_2, then we will show that it is a MOC if and only if both G_1 and G_2 are of the same type (i.e., one is a location-scale transform of the other) as some non-degenerate symmetric distribution G with finite second moment. Moreover, $\kappa_{G_1^{-1},G_2^{-1}} = \kappa_{G^{-1}}$ for the triplet (G_1, G_2, G). This class includes various popular MOCs such as *Spearman's rho* and *Blomqvist's beta*.

For a distribution G satisfying the conditions above, let $\mathcal{K}_G$ be the set of all $\kappa_{G^{-1}}$-compatible $d \times d$ matrices. Then we will show that $\mathcal{K}_G$ is convex and there exists the bounds

$$\mathcal{P}_d^{\mathrm{B}}(1/2) \subseteq \mathcal{K}_G \subseteq \mathcal{P}_d$$

where $\mathcal{P}_d$ is the set of all $d \times d$ correlation matrices and $\mathcal{P}_d^{\mathrm{B}}(1/2)$ is the set of all $d \times d$ correlation matrices with margins Bern(1/2), also characterized by $\mathcal{P}_d^{\mathrm{B}}(1/2) = \mathrm{conv}\{\boldsymbol{c}\boldsymbol{c}^\top : \boldsymbol{c} \in \{-1, 1\}^d\}$.

3. References:

Scarsini, M. (1984) On measures of concordance, *Stochastica*, **8**(3), 201-218.

Hofert, M. and Koike, T. (2019). Compatibility and attainability of matrices of correlation-based measures of concordance. *ASTIN Bulletin*, 1-34.

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