Nonlinear Leverage Effects in Asset Returns: Evidence from American and Japanese Stock Markets

Asahi Ushio*
Faculty of Scinence and Engineering, Keio University
and
Kenichiro McAlinn†
Department of Statistical Science, Duke University
and
Teruo Nakatsuma‡
Faculty of Economics, Keio University

August 12, 2015

Abstract

We develop a new structure to model the correlation between an asset's return and its volatility in stochastic volatility models. This correlation is often referred to as the leverage effect in the literature and is considered to be negative so that a negative shock in a stock price increases its volatility. While this relation is intuitive and is followed by a plethora of economic reasoning, empirical evidence suggests that most individual stocks have zero correlation and thus no leverage effect. Therein lies the leverage effect puzzle. In this paper, we develop a nonlinear generalization of the leverage effect, or rather the leverage function, within a stochastic volatility setting. We adopt Hermite polynomials as the orthogonal basis of the leverage function and estimate the parameters via particle learning, a sequential Monte Carlo method when the sufficient statistics of the parameters are known. However, for the model of interest and in most complex nonlineiar/non-Gaussian models, the sufficient statistics are unknown. For this reason, we develop a novel and flexible particle learning algorithm using auxiliary variables. Examining 682 stocks that composite the S&P500, NASDAQ, and Nikkei 225, we find four thirds of the stocks to exhibit complex volatility structures. We further the analysis by examining whether there are clear traits in which stock have more complex volatility structures. We find evidence that country and few sectors to have an effect on the volatility structure.

Keywords: Leverage Effect, Particle Learning, Stochastic Volatility Model.

*Student, Faculty of Science and Engineering, Keio University, 2-15-45 Mita, Minato-ku, Tokyo, 108-8345, Japan
†Graduate Student, Department of Statistical Science, Duke University, Durham, NC 27708.
‡Professor, Faculty of Economics, Keio University, 2-15-45 Mita, Minato-ku, Tokyo, 108-8345, Japan.

1 INTRODUCTION

In the field of finance and economics, volatility of financial assets has been investigated with great scrutiny to further understand the mechanics and structure of price movement. One aspect of volatility that has gathered especial interest recently is the leverage effect. The term leverage refers to an economic interpretation given by Black (1976) and Christie (1982). The idea is that, when a asset's price declines, the company's relative debt increases and thus is "leveraged", making the company relatively riskier and therefore more volatile. The different interpretations have been investigated and compared in Bekaert and Wu (2000), for example. In this paper, we estimate the SV model with nonlinear leverage function by particle learning with auxiliary variables. We will define the SV model with nonlinear leverage function in Section 2 and particle learning with auxiliary variables in Section 3. Section 4 will present the empirical study where we apply our model to daily returns of all stocks that compose the NASDAQ100, S&P500, and Nikkei 225.

2 STOCHASTIC VOLATILITY MODEL WITH LEVERAGE FUNCTION

The SV model is a state space model with observation noise ε_t and state noise η_t. Both ε_t and η_t are mutually and serially independent in this model represented by the off diagonal elements in the covariance matrix being zero. Yu (2005) compares two types of asymmetric SV models in order to reflect the leverage effect in the model. The widely used asymmetric SV model is the SV model with assumption of correlation between ε_t and η_t, such as

$$\begin{cases} y_t = \exp\left(\frac{x_t}{2}\right)\varepsilon_t, \\ x_{t+1} = \mu + \beta x_t + \eta_t, \end{cases} \quad \begin{bmatrix}\varepsilon_t \\ \eta_t\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\tau \\ \rho\tau & \tau^2\end{bmatrix}\right), \tag{2.1}$$

where ρ is the correlation between the observation noise ε_t and state noise η_t. This model allows for a contemporaneous dependence in the variance. Then, setting $u_t \sim \mathcal{N}(0,1)$, the state noise η_t can be rewritten as

$$\eta_t = \rho\tau\varepsilon_t + \sqrt{1-\rho^2}\tau u_t \tag{2.2}$$

hence, the asymmetric SV model (2.1) can be rewritten as a Gaussian nonlinear state space model with uncorrelated observation noise u_t and state noise ε_t such as

$$\begin{cases} y_t = \exp\left(\frac{x_t}{2}\right)\varepsilon_t, \\ x_{t+1} = \mu + \beta x_t + \varphi\varepsilon_t + \omega u_t, \end{cases} \quad \begin{bmatrix}\varepsilon_t \\ u_t\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right), \tag{2.3}$$

where $\varphi = \rho\tau$, and $\omega = \sqrt{1-\rho^2}\tau$. This model assumes linear correlation, yet the assumption is not founded in evidence but in convenience. Therefore we extend the model to include nonlinear leverage. Following Hansen, Huang and Shek (2011), we use Hermite polynomials to construct the leverage function. Then the k-th order Hermite polynomial $H_k(z)$ is defined as

$$H_k(z) = (-1)^k \exp\left(\frac{z^2}{2}\right)\frac{d^k}{dz^k}\exp\left(-\frac{z^2}{2}\right). \tag{2.4}$$

The Hermite polynomial has many desirable factors such as:

1. Expectation: When z follows a random distribution with zero expected value and unit variance, the expected value of the Hermite polynomial is equal to zero for any k. In other words,

$$E(H_k(z)) = \int_{-\infty}^{\infty} H_k(z)p(z)dz = 0,\ \ z \sim N(0,1)$$

where $p(z)$ is the probability density function of the standard normal distribution.

2. Orthogonality: The Hermite polynomial is orthogonal with respect to the weight function $\phi(x) \propto \exp(-\frac{x^2}{2})$ such as $p(z)$. So thus

$$E(H_j(z)H_k(z)) = \int_{-\infty}^{\infty} H_j(z)H_k(z)\phi(z)dz = \begin{cases} n! & (k = j) \\ 0 & (k \neq j) \end{cases}$$

Given the above properties of the Hermite polynomials, they can be orthogonal basis of the Hilbert space of leverage functions such that

$$\int_{-\infty}^{\infty} |\ell(z)|^2 \phi(z)dz < \infty. \tag{2.5}$$

Thus we can approximate $\ell(\varepsilon_t)$ by

$$\ell_k(\varepsilon_t) \equiv \varphi_1 H_1(\varepsilon_t) + \cdots + \varphi_k H_k(\varepsilon_t), \tag{2.6}$$

when k is sufficiently large.

Finally, the SV model with k-th nonlinear leverage function is defined as

$$\begin{cases} y_t = \exp\left(\frac{x_t}{2}\right)\varepsilon_t, \\ x_{t+1} = \mu + \beta x_t + \ell_k(\varepsilon_t) + \omega u_t, \end{cases} \quad \begin{bmatrix} \varepsilon_t \\ u_t \end{bmatrix} \sim \mathscr{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right), \tag{2.7}$$

3 PATICLE LEARNING WITH AUXILIARY VARIABLES

Generally in econometrics, parameters of the model are usually unknown and are of interest. When a state space model depends on unknown but fixed parameters θ,

$$\begin{cases} y_t \sim p(y_t|x_t, \theta) \\ x_t \sim p(x_t|x_{t-1}, \theta) \end{cases} \tag{3.1}$$

we need to evaluate the posterior distribution $p(\theta|y_{1:t})$ given the observations $y_{1:t}$. In the framework of particle filtering, $p(\theta|y_{1:t})$ is sequentially updated as new observations arrive. This is called particle learning. The particle learning algorithm is defined as follows. Let $\{z_t^{(i)} = (x_t^{(i)}, \theta_t^{(i)})\}_{i=1}^N$ and $\{\tilde{z}_t^{(i)} = (\tilde{x}_t^{(i)}, \tilde{\theta}_t^{(i)})\}_{i=1}^N$ denote particles jointly generated from $p(x_t, \theta|y_{1:t-1})$ and $p(x_t, \theta|y_{1:t})$ respectively. Then the particle approximation of the Bayesian learning process is given by

$$p(z_t|y_{1:t-1}) \simeq \frac{1}{N}\sum_{i=1}^N p(z_t|\tilde{z}_{t-1}^{(i)}), \tag{3.2}$$

$$p(z_t|y_{1:t}) \simeq \sum_{i=1}^N \omega_t^{(i)}\delta(z_t - z_t^{(i)}), \quad \omega_t^{(i)} = \frac{p(y_t|z_t^{(i)})}{\sum_{i=1}^N p(y_t|z_t^{(i)})}. \tag{3.3}$$

The particle learning algorithm by Carvalho, Johannes, Lopes and Polson (2010) learns the parameters by updating the sufficient statistics of the parameter distribution, denoted by s_t, via the recursion map, $\mathscr{S}(\cdot)$.In this paper, we use particle learning to estimate the SV model with nonlinear leverage function. In the case of adaption to the SV model, the parameters of the SV model are defined as $\theta = (\gamma, \omega^2)$ where $\gamma' = [\mu \;\; \beta \;\; \varphi_1 \;\; \dots \;\; \varphi_k]$. In this context, θ is sampled from the conditional posterior

$$\gamma|\omega^2 \sim N_{k+2}(b_t, \omega^2 A_t^{-1}), \;\; \omega^2 \sim inverseGamma(c_t, d_t), \tag{3.4}$$

where the sufficient statistics $s_t = (A_t, b_t, c_t, d_t)$ are computed by the following recursion

$$A_t = A_{t-1} + Z_t Z_t', \quad b_t = A_t^{-1}(A_{t-1}b_{t-1} + x_t Zt),\ c_t = c_{t-1} + \frac{1}{2}, \quad d_t = d_{t-1} + \frac{(x_t - Z_t' b_{t-1})^2}{2(1 + Z_t' A_{t-1}^{-1} Z_t)}, \tag{3.5}$$

where $Z_t' = [1 \;\; x_{t-1} \;\; H_1(\varepsilon_{t-1}) \;\; \ldots \;\; (k!)^{-1/2} \;\; H_k(\varepsilon_{t-1})]$. While the above method is useful, when the sufficient statistic is unknown or hard to calculate, it becomes extremely difficult to compute the unknown parameters. According to Pitt and Shephard (1999), those filtering method based on SIR provides good estimations of state variables and parameters where the model is a good approximation to the data and the conditional densities $p(y_t|x_t)$ are reasonably flat in x_t. They point out two weaknesses of the algorithm. Firstly, when there is aberrant observation, the SIR method cannot adopt precisely so the state variables would underestimate even when N is large. In this case, the variability of observation values would be increased and that reduces the accuracy of resampling because it is based on weights by the likelihood. Secondly, the degeneration of particles are a general problem which occurs when the likelihood is similar at each time. It causes poor tail representation in the prediction density because the particles placed on the tails are resampled by similarly low weights each time that the weights increasingly reduces to zero. As a result, the posterior degenerates to a few points. For those problems, we adopt the auxiliary filter by Pitt and Shephard (1999) in our particle learning algorithm. This method divides the resampling procedure into two resampling processes, which at first resamples like usual and then resamples via auxiliary variable k. The first resampling is defined as

$$p(\tilde{x}_t|y_{1:t}) \simeq \sum_{k=1}^{N} \tilde{\omega}_t^{(k)} \delta(x_t - x_t^{(k)}), \quad \tilde{\omega}_t^{(k)} = \frac{p(y_t|x_t^{(k)})}{\sum_{k=1}^{N} p(y_t|x_t^{(k)})} \tag{3.6}$$

where k is an index and $x_t^{(k)}$ is the mean, the mode, or some other likely value associated with the density of $p(x_t|x_{t-1})$. Weights $\tilde{\omega}_t^{(k)}$ are the first stage weights. The second resampling is defined as

$$p(x_t|y_{1:t}) \simeq \sum_{i=1}^{N} \omega_t^{(i)} \delta(x_t - x_t^{(i)}), \quad \omega_t^{(i)} = \frac{\dfrac{p(y_t|x_t^{(i)})}{\sum_{i=1}^{N} p(y_t|x_t^{(i)})}}{\dfrac{p(y_t|x_t^{(k)})}{\sum_{k=1}^{N} p(y_t|x_t^{(k)})}} \tag{3.7}$$

where weights $\omega_t^{(k)}$ are the second stage weights. This divided process softens the effect of outliers because the second stage resampling would be much less variable than the original SIR resampling. Moreover the first stage resampling is based on the current observation value so that good particles are propagated forward. The algorithm of particle learning with auxiliary variable k is summarized as follows.

ALGORITHM: PARTICLE LEARNING WITH AUXILIARY VARIABLES

Step 0: Sample the starting values of N particles $\{z_0^{(i)}\}_{i=1}^N$ from $p(z_0)$.

Step 1: Resample $\{\tilde{z}_{t+1}^{(i)}\}_{i=1}^N$ from $z_t^{(i)} = (x_t, s_t, \theta)^{(i)}$ with weights $\tilde{\omega}_{t+1} \propto p(y_{t+1}|z_t^{(k)})$ such that $\sum_{i=1}^N \tilde{\omega}_t^{(i)} = 1$.

Step 2: Propagate $\tilde{x}_t^{(i)}$ to $\tilde{x}_{t+1}^{(i)}$ via $p(x_{t+1}|\tilde{z}_{t+1}^{(i)}, y_{t+1})$.

Step 3: Resample $\{x_{t+1}^{(i)}\}_{i=1}^N$ from $\{\tilde{x}_{t+1}^{(i)}\}_{i=1}^N$ with weights $\omega_{t+1} \propto \dfrac{p(y_{t+1}|\tilde{z}_t^{(i)})}{p(y_{t+1}|\tilde{z}_t^{(k)})}$

such that $\sum_{i=1}^N \omega_t^{(i)} = 1$.

Step 4: Update sufficient statistics $s_{t+1}^{(i)} = \mathscr{S}(s_t^{(i)}, x_{t+1}^{(i)}, y_{t+1})$.

Step 5. Sample $\theta^{(i)}$ from $p(\theta|s_{t+1}^{(i)})$.

Finally, we adopt the particle filter with auxiliary variable to the SV model with nonlinear leverage function. Updating sufficient statistics s_t and sampling parameters θ are calculated following (3.4) and (3.5).

For model selection, in the framework of particle filtering, we select models based on marginal likelihoods and this can be calculated directly in the filtering procedure from each particle's likelihood. For example, for model M, the marginal likelihood $p(y_{1:t}|M)$ is

$$p(y_{1:t}|M) = \prod_{j=1}^{t} p(y_j|y_{1:j-1}, M).$$

Then, it can be approximated by N particles such as

$$p(y_{1:t}|M) \simeq \frac{1}{N} \prod_{j=1}^{t} p^N(y_t|y_{t-1}^{(i)}, M), \tag{3.8}$$

where

$$p(y_t|y_{t-1}, M) \simeq \sum_{i=1}^{N} p(y_t|y_{t-1}^{(i)}, M) = p^N(y_t|y_{1:t-1}, M).$$

4 EMPIRICAL STUDY

4.1 Results from Particle Learning

In the empirical study, we use data from the major equity markets; Nikkei225, S&P500, and NASDAQ100 in order to compare the difference in volatility structure between country and markets. We use daily stock closing prices of all stocks that compose the three indexes from the beginning of 2004 to the end of 2013. [1]. Since some of the stocks weren't listed for the time period, they were omitted from the analysis. A total of 683 stocks were analyzed with 198 stocks from Nikkei225, 417 stocks from S&P500, and 68 stocks from NASDAQ100. We use an asset's return such as,

$$y_{i,t} = 100 \times (\ln R_{i,t} - \ln R_{i,t-1}),$$

where $R_{i,t}$ is a closing price of stock i at term t. We applied each stock return data to the SV model with $k = 1 \sim 6th$ leverage function. They were compared based on their cumulative log marginal likelihood with a learning period of two years and eight years of data. The model with the highest cumulative log marginal likelihood was selected as the optimal model.

Figure 1(a) shows the number of optimal orders. As seen in figure 1(a), the $2nd$ order leverage function is selected by the most. Orders of above two comprise 75% of all stocks, strongly suggesting that the leverage is in fact nonlinear. $2th$ leverage function is selected by the most stocks in all and more than $4th$ leverage functions are much less than $1 \sim 3rd$ leverage function. We can see that, most stocks are classified in $1st \sim 3rd$ leverage function. Figure 1(b),3 are the number of optimal orders by country and sector. In figure 1(b), we see that $2nd$ order leverage function is selected by the most stocks in both countries. There is an apparent difference between the number of orders selected between Japan and the U.S.A. It is obvious that $1st$ order leverage functions are supported proportionally more in Japanese stocks then American.

[1]Data was collected from 'Bloomberg'.

We further our analysis by examining the sectors as seen in figure 3. Energy is the only sector which supports $1st$ order leverage function. Except for Energy, $2nd$ order leverage function is the most selected order. Focusing on $1st \sim 3rd$ leverage function's number, four sectors (Consumer Discretionary, Financials, Materials, Utilities) are the type of that $1st$ leverage function is more than $3rd$ leverage function. On the other hand, three sectors (Consumer Staples, Health Care, Communications) are the type of that $3rd$ leverage function is more than $1st$ leverage function. Other sector (Industrials, Technology) are the type of that the number of $1st$ leverage function and $3rd$ leverage function is almost the same. Sectors of the first type can be said to that the rate of increase depends on past observation noise is smaller than sectors of the second type. The difference among sectors can be thought as each sector's volatility property. Although each sector seems to have some characteristic, there is no clear pattern between each sector.

Table 1 shows how many stocks have a positive or negative sign on the first order. We see a good number of stocks have a positive sign, suggesting that there is a positive correlation between asset returns and volatility, which is very counter intuitive. Figure 2, for example, describe the $1 \sim 3rd$ and $4 \sim 6th$ order leverage function of U.S and Japan's highest order. This shows the complexity of the leverage function never shown before. While $1st$ order function increases monotonically as the observation noise decreases, most other leverage functions have a nonlinear relationship. Instead of the leverage effect, these leverage functions, except for the $1st$ order, show an effect which decreases volatility when observation noise increases. Though the tails of higher order leverage functions are unstable, it is stable between credible interval of the observation noise.

4.2 Further Analysis

From the results, we can read off some patterns. However, the effects of country and sector is unclear without further inspection. For this we employ a ordered probit model to analyze the effects such as

$$y_i^* = constant + \boldsymbol{\beta} \boldsymbol{x}_i + u_i, \ \ u_i \sim \mathcal{N}(0,1) \tag{4.1}$$

$$y_i = \begin{cases} 1 \ (\mu_0 \leq y_i^* < \mu_1), \\ 2 \ (\mu_1 \leq y_i^* < \mu_2), \\ 3 \ (\mu_2 \leq y_i^* < \mu_3), & \boldsymbol{x}_i' = [x_{i,1}, ..., x_{i,12}] \\ 4 \ (\mu_3 \leq y_i^* < \mu_4), & \boldsymbol{\beta} = [\beta_1, ..., \beta_{12}] \\ 5 \ (\mu_4 \leq y_i^* < \mu_5), \\ 6 \ (\mu_5 \leq y_i^* < \mu_6), \end{cases}$$

where y_i^* is a latent continuous variable which can takes one of six order categories y_i classified by bins $\mu_0 \sim \mu_5$ where $\mu_0 = 0$ and $\mu_6 = \infty$. $\boldsymbol{x}_i$ is factor loading vector and $\boldsymbol{\beta}$ is factor vector which are defined below.

$$\begin{aligned} x_{i,1} \ &: \ \text{Country (if } y_i \text{ is Japanese stock, } x_{i,1} = 1. \text{ Else } x_{i,1} = 0. \text{)} \\ x_{i,2} \sim x_{i,10} \ &: \ \text{Sector (if } y_i \text{ is sector } k, \ x_{i,k-1} = 1. \text{)} \\ x_{i,11} \ &: \ \text{Volatility} \\ x_{i,12} \ &: \ \text{Trade Volumes} \end{aligned}$$

The finance sector has been omitted due to identification so the results are relative to the finance sector. Volatility is the average volatility of the stock and Trade Volume is the average number of trade volume for the stock during that period. In general, the regression model is estimated by minimizing the likelihood. For each i, the likelihood is

$$\begin{aligned}\Pr(y_i = j|\beta, \mu_{k:k=1\sim5}) &= \Pr(\mu_{j-1} < y_i^* \le \mu_j) \\ &= \Pr(\mu_{j-1} - \beta x_i < y_i^* - \beta x_i \le \mu_j - \beta x_i) \\ &= \Phi(\mu_j - \beta x_i) - \Phi(\mu_{j-1} - \beta x_i) \end{aligned} \tag{4.2}$$

where $\Pr(\cdot)$ is probability, $\Phi(a)$ is the probability density function of a normal distribution at point a. For our analysis, following James and Siddhartha (1993), we use Gibbs sampling to estimate the posterior. The posterior of y_i^* given $\beta, \mu_{k:k=1\sim5}, y_i$ is

$$y_i^*|\boldsymbol{\beta}, \mu_{k:k=1\sim5}, y_i \sim TN_{[\mu_{j-1},\mu_j)}(\boldsymbol{\beta x}_i, 1) \tag{4.3}$$

where $TN_{[a,b)}(\mu, \sigma^2)$ is a normal distribution with mean μ and variance σ^2 which is truncated at the left by a and right by b. Then the posterior of β given $y_i^*, \mu_{k:k=1\sim5}, y_i$ is

$$\boldsymbol{\beta}|\mu_{k:k=1\sim5}, y_i, y_i^* \sim \mathcal{N}((\boldsymbol{x}_i^{'}\boldsymbol{x}_i)^{-1}\boldsymbol{x}^{'}y_i^*, (\boldsymbol{x}_i^{'}\boldsymbol{x}_i)^{-1}) \tag{4.4}$$

and the posterior of $\mu_{k:k=1\sim5}$ given $y_i^*, y_i, \boldsymbol{\beta}$ is

$$\mu_{k:k=1\sim5}|\boldsymbol{\beta}, y_i, y_i^* \sim U[a_k, b_k] \tag{4.5}$$

where

$$a_k = \max\left\{\alpha_{k-1}, \max_{i:y_i=k} y_i^*\right\}, \quad b_k = \max\left\{\alpha_{k+1}, \min_{i:y_i=k+1} y_i^*\right\}.$$

For the initial condition of β, we used the MLE of β, and estimated the parameters' distribution via Gibbs sampler from conditional posterior (4.3), (4.4), and (4.5). The results of the ordered probit model is shown in figure 4 and table 2. Figure 4 is the histogram of each parameter $\beta_1 \sim \beta_{12}$ and α. Table 2 shows posterior means and 95% credible interval (CI) of each parameter. From this we can see that country is the only significant variable. The results of gibbs sampler estimation of ordered probit regression model are shown in figure 4 and table 2. Figure 4 is the histogram of each parameter $\beta_1 \sim \beta_{12}$ and $constant$. Table 2 shows posterior means and 95% credible intarval (CI) of each parameter. With Country, the results show that there is a bias on the leverage function between Japanese and American. In this case, figure 4 shows American stocks to have a positive effect on the leverage function order. Regarding Sector, we can see that a few sectors, such as Technology, Health Care, Consumer Staples, have a possitive effect on the order. Figure 4, also implies that volatility has a slight negative effect on the leverage function so the order of leverage function slightly decrease when the volatility increase. However, its posterior mean is about zero so the effect would be trivial. Moreover, the liquidity factor's distribution seems have heavy tail and the posterior mean is almost zero so it can be said that liquidity does not have any effects on the leverage function.

5 CONCLUSION

In specific, 2*nd* order leverage function is most supported in most and implies that simple linear leverage effect is inadequate. Moreover, through ordered probit regression, we find that the leverage order is effected by the country, market, and sector. In particlar, we find that country is the most effective factor determining the leverage order. We found that American stocks support higher order leverage functions than Japanese stocks and especially stocks which are components of NASDAQ100 have higher leverage functions. As mentioned in section 4, we believe this is because NASDAQ100 is a market for new companies and sectors. When the results are compared by sector, the technology sector tends to suppprt higher order leverage than any other sector.

Looking at the difference between S&P500 and Nikkei225, we see that the proportion of 3*rd* order leverage function is higher in the US compared to Japanese. From those differences, we can infer that the leverage effect is more complex in the US than Japan. With regards to sector, we find that the energy sector to have lower order leverage and the technology sector to have higher order leverage. Through these results we find strong evidence that the leverage effect does exist in individual stocks, but in very complex structures that can be overlooked using simple models.

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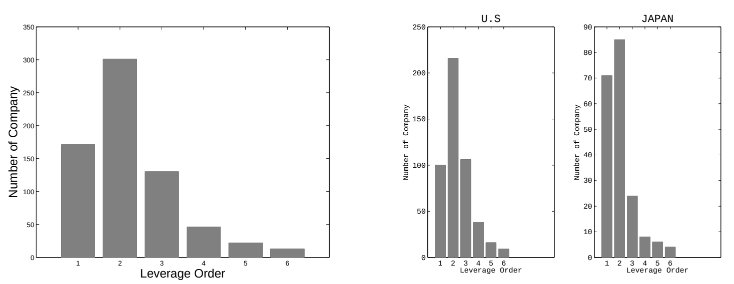
Table 1: Sign of the highest order parameter (positive or negative)

Order of	Nikkei225		S&P500		NASDAQ100	
Leverage	Positive	Negative	Positive	Negative	Positive	Negative
1	24	47	3	93	0	4
2	18	67	3	177	0	36
3	6	18	62	27	10	7
4	4	4	8	22	0	8
5	1	5	6	8	1	1
6	1	3	5	3	1	0

Table 2: Posterior Mean & 95% Creadible Interval.

Parameter	constant	Country	Volatility	Liquidity
Posterior mean	0.176	0.318	-0.040	-1.894
CI	[-0.292 0.644]	[0.107 0.532]	[-0.237 0.157]	[-17.341 13.547]
	Sector			
Parameter	Health Care	Industrials	Technology	Materials
Posterior mean	0.205	-0.059	0.253	0.044
CI	[-0.135 0.544]	[-0.380 0.267]	[-0.027 0.535]	[-0.284 0.377]
	Sector			
Parameter	Energy	Communications	Utilities	Consumer Discretionry
Posterior mean	-0.219	-0.182	0.015	-0.016
CI	[-0.648 0.207]	[-0.980 0.619]	[-0.363 0.399]	[-0.320 0.286]
	Sector			
Parameter	Consumer Staples			
Parameter mean	0.159			
CI	[-0.187 0.503]			



(a) The selected model order in all stocks (b) The selected model order by country

Figure 1: The selected model order

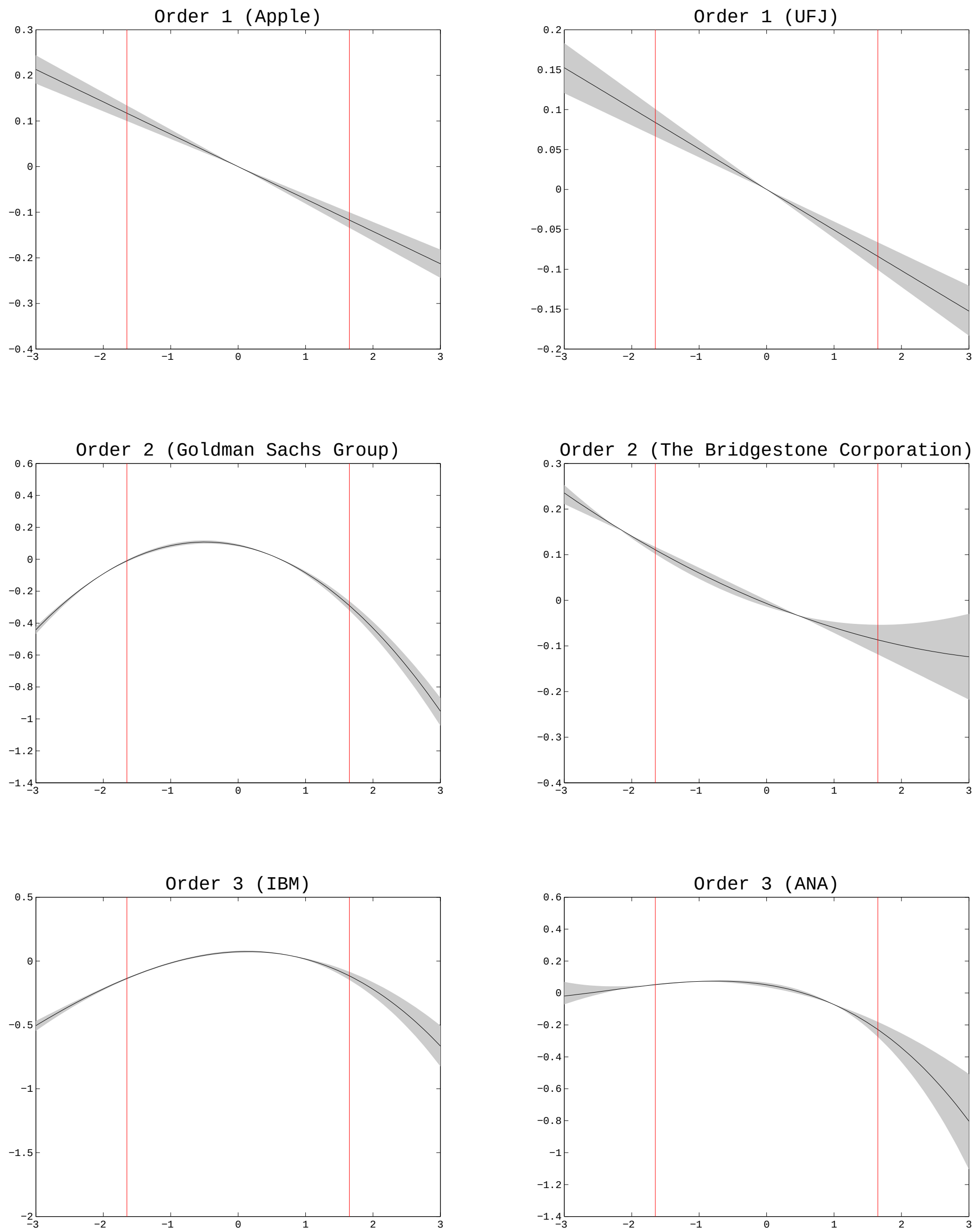


Figure 2: $1 \sim 3rd$ leverage function. The gray area and vertical lines are 95% credible interval of the function and observation noise. Left:U.S, Right:Japan.

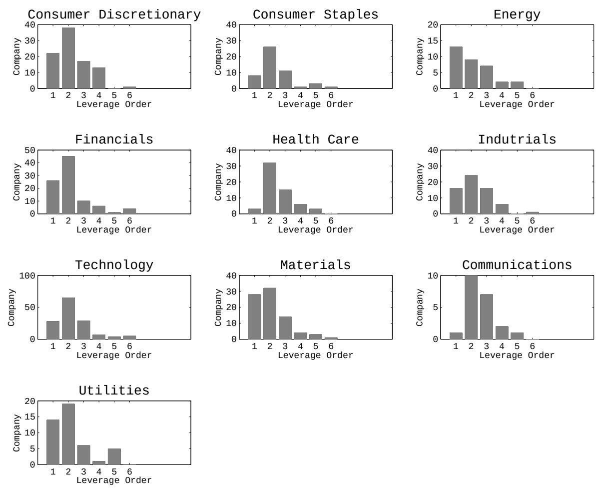


Figure 3: The selected model order by sector

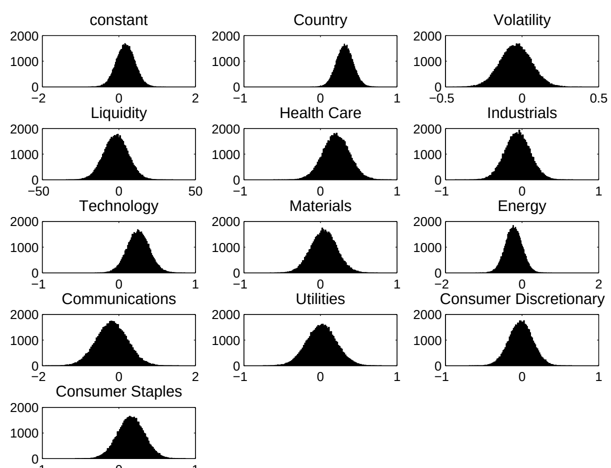


Figure 4: Parameter histogram given by results of gibbs sampler